Lecture 5 — ∞-Categories, Enriched Categories, Simplicial Localization, and Framings

∞-Groupoids

Definition 1 (∞-groupoid)**.** An ∞**-groupoid** consists of:

- for each $n \in \mathbb{N}$, a set of n**-morphisms**;
- for each $n > 0$, maps $s_n, t_n \colon n\text{-Mor} \to (n-1)\text{-Mor}$;
- for each $n > 0$ maps $i_n \colon n\text{-Mor} \to n\text{-Mor}$ such that $s_{n+1} i_n f = t_{n+1} f, t_{n+1} i_n f = s_{n+1} f$;
- for each $n \geq 0$, maps $\varepsilon_n \colon n\text{-Mor} \to n+1\text{-Mor}$;
- for each $n > 0$, a **composition map** $\circ_n \colon n\text{-Mor} \times_{s_n, t_n} n\text{-Mor} \to n\text{-Mor}$;

such that **every** n**-morphism is invertible**.

Definition 2 (Fundamental ∞-groupoid)**.** For $X \in \text{Top}$, define the **fundamental** ∞**-groupoid** $\Pi_\infty X$ as follows:

- 0-morphisms $= X$;
- 1-morphisms $= \text{Top}(I, X)$ (paths in X);
- 2-morphisms between $\gamma \to \eta$: homotopies of paths, i.e. elements of $\text{Top}(I^2, X)$ relative to the boundary.

In general, n**-morphisms are homotopies between** $(n-1)$**-morphisms with the same endpoints**.

Lemma 3. *The following functors are all Quillen equivalences:*

$$\begin{tikzcd}
\text{SSet} \arrow[rr, shift left, "|-|"] \arrow[dr, shift left] & & \text{Top} \arrow[ll, shift left, "\text{Sing}"] \arrow[dl, shift left] \\
& \infty - Grpd \arrow[ul, shift left, "N"] \arrow[ur, shift left, "\Pi_\infty"'] &
\end{tikzcd}$$

Enriched Categories

Definition 4 (Enriched category)**.** Let $\mathcal{V}$ be a **monoidal category**:

$$\big(\otimes\colon \mathcal{V}\times\mathcal{V}\to\mathcal{V},\ I\in\operatorname{Ob}\mathcal{V},\ \alpha_{a,b,c}\colon (a\otimes b)\otimes c\xrightarrow{\sim} a\otimes(b\otimes c),\ I\otimes a\simeq a\simeq a\otimes I\big).$$

A **small $\mathcal{V}$-enriched category** C consists of:

- a set $\operatorname{Ob}(C)$ of objects;
- for every $(a,b)\in\operatorname{Ob}(C)^2$, an object $C(a,b)\in\operatorname{Ob}(\mathcal{V})$ (the **hom-object**);
- for every $a,b,c\in\operatorname{Ob}(C)$, a **composition map** $\circ_{a,b,c}\colon C(b,c)\otimes C(a,b)\to C(a,c)$;
- for every $a\in\operatorname{Ob}C$, an **identity map** $j_a\colon I\to C(a,a)$;

such that composition is associative and j_a is a two-sided identity (in the appropriate diagrammatic sense).

Example 5.

1. Every locally small category C is Set-enriched.
2. $C=\mathrm{Vect}_k$: $\mathrm{Hom}_{\mathrm{Vect}_k}(a,b)\in\mathrm{Vect}_k$, so C is Vect_k-enriched.
3. $C=\mathrm{Ch}(k)$: if $\mathrm{Hom}(a,b)=$ chain maps of degree 0, then C is Vect_k-enriched; if $\mathrm{Hom}(a,b)=$ maps of all degrees $+d$, then C is $\mathrm{Ch}(k)$-enriched.
4. Grp is *not* Grp-enriched, but Ab is Ab-enriched.

Definition 6 (Groupoid and $(2,1)$-category)**.** A **groupoid** is a small category in which every morphism is an isomorphism.

A **(strict)** $(2,1)$**-category** (equivalently, a category enriched over Grpd) is a category C with:

- a set $\operatorname{Ob}(C)$ of objects;
- for every a,b, a set $1\text{-Mor}(a,b)$ of 1-morphisms ($=\operatorname{Ob}(G)$ for G a groupoid);
- for every $f,g\in 1\text{-Mor}(a,b)$, a set $2\text{-Mor}(f,g)$ of **invertible** 2-morphisms.

Example 7 (Bimodules)**.** $\operatorname{Ob}(C)=\{$not necessarily commutative rings$\}$, $1\text{-Mor}(a,b)=\{$left b, right a-modules (bimodules)$\}$, 2-Mor $=$ isomorphisms of such modules.

Definition 8 ((n,r)-category)**.** An (n,r)**-category** has k-morphisms for $k=0,\dots,n$ (objects to n-morphisms), with all k-morphisms for $k>r$ being invertible.

Definition 9 ($(\infty,1)$-category)**.** An $(\infty,1)$**-category** is (for our purposes) a category enriched over ∞-groupoids, or equivalently over Top or SSet.

Goal: Given C a model category, we want to produce a canonical $(\infty,1)$-category $\widetilde{C}$.

From Categories to Simplicial Categories

Definition 10 (Comonad from an adjunction)**.** Given an adjunction

$$\begin{tikzcd} C \arrow[r, bend left, "U"] & D \arrow[l, bend left, "F"] \end{tikzcd} \qquad \eta\colon \mathrm{Id}_D \to UF, \quad \varepsilon\colon FU \to \mathrm{Id}_C,$$

define:

$$G = FU\colon C \to C, \qquad \delta\colon G \to G \circ G, \quad \delta = F \circ \eta \circ U\colon FU \to FUFU.$$

The triple (G, ε, δ) is a **comonad** on C.

For $X \in \operatorname{Ob} C$, set $J_i(X) = G^{i+1}(X)$ and define:

$$\begin{aligned} d_k(X)\colon G^{n+1}(X) \to G^n(X), &\quad d_k = G^{n-k} \circ \varepsilon \circ G^k(X), \\ s_k(X)\colon G^n(X) \to G^{n+1}(X), &\quad s_k(X) = G^{n-k-1} \circ \delta \circ G^k(X). \end{aligned}$$

These are **face** and **degeneracy maps**, yielding a functor

$$J\colon C \longrightarrow \mathrm{Fun}(\Delta^{\mathrm{op}}, C).$$

Definition 11. Let $U\colon \mathrm{Cat} \to \mathrm{Graph}$ be the forgetful functor sending a category to its underlying graph, and $F\colon \mathrm{Graph} \to \mathrm{Cat}$ its left adjoint (free category). We get a functor

$$J\colon \mathrm{Cat} \longrightarrow \mathrm{SCat} \quad \text{(simplicial categories)}.$$

Remark. *A simplicial category has a* simplicial set *of objects and a* simplicial set *of morphisms.*

Lemma 12. *The functor* $J\colon \mathrm{Cat} \to \mathrm{SCat}$ *takes values in simplicial categories with a* discrete *simplicial set of objects, i.e. in* SSet*-enriched categories.*

Simplicial Localization

Definition 13 (Simplicial localization)**.** Given a category with weak equivalences (C, W), its **simplicial localization** is the simplicial category

$$L_n(C, W) \ := \ J_n C\big[J_n W^{-1}\big].$$

This again has a discrete simplicial set of objects and is therefore a SSet-enriched category, i.e. an $(\infty, 1)$-category.

Theorem 14. *All* $(\infty, 1)$*-categories (up to equivalence) arise as simplicial localizations of categories with weak equivalences.*

Simplicial localization in model categories

Let C be a model category. Since Δ^{op} is a Reedy category, $\mathrm{Fun}(\Delta^{\mathrm{op}}, C) = C^{\Delta^{\mathrm{op}}}$ carries the **Reedy model structure**.

The evaluation functor

$$\mathrm{Ev}_0\colon C^{\Delta^{\mathrm{op}}} \to C, \quad X_\bullet \mapsto X_0$$

has **both** a left and a right adjoint:

- Left adjoint: $\mathrm{Const} \dashv \mathrm{Ev}_0$ (constant diagrams);
- Right adjoint: $\mathrm{Ev}_0 \dashv P$, where

$$P\colon C \to C^{\Delta^{\mathrm{op}}}, \quad X \mapsto \big([n] \mapsto X^{n+1}\big).$$

This yields a natural morphism $f\colon \mathrm{Const}\, X \to PX$ in $C^{\Delta^{\mathrm{op}}}$.

Definition 15 (Simplicial frame and right framing)**.**

1. A **simplicial frame on** X is a factorization

$$\begin{tikzcd}
\mathrm{Const}\, X \arrow[rr, "f"] \arrow[dr, "W"'] & & PX \\
& \tilde{S}X \arrow[ur, "\mathrm{Fib}"'] &
\end{tikzcd}$$

   where $f \in W \cap \mathrm{Cof}$ (i.e. f is a trivial cofibration).

2. A **right framing** on C is a functor $S\colon C \to C^{\Delta^{\mathrm{op}}}$ such that
   - $\mathrm{Ev}_0(SX) = X$ for all X, and
   - SX is a simplicial frame on X.

Hom in the Simplicial Localization

Theorem 16. *Let C be a model category, $X \in C_c$ (cofibrant), $A \in C_f$ (fibrant). Define the* ***simplicial hom-set***

$$\mathrm{Hom}(X, A)_\bullet \ := \ \mathrm{Hom}_{C^{\Delta^{\mathrm{op}}}}(\mathrm{Const}\, X,\ SA),$$

equivalently,

$$\mathrm{Hom}(X, A)_n \ = \ \mathrm{Hom}_C\big(X,\ (SA)_n\big),$$

where S is any right framing on C. Then:

$$\mathrm{Hom}(X, A)_\bullet \textit{ is the hom-set in the simplicial localized category } L(C, W).$$

In other words, $\mathrm{Hom}(X, A)_\bullet$ *computes the mapping space in the* $(\infty, 1)$*-category associated to* (C, W)*.*